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1. **Introduction.** The differential geometry of a surface (spread of two dimensions) owes much of its development upon the notion of the curvature of the surface introduced by Gauss; similarly, the differential geometry of a spread of n dimensions bases much of its subject matter on the notion of the curvature of such a spread as introduced by Riemann. This important concept of the curvature of a space according to Riemann may be summarized as follows: In a space V_n defined by the totality of points given by the values of n real independent variables $x_1, x_2, \ldots, x_n$, we have for the element of length or first fundamental form, the quadratic differential form

$$ds^2 = \sum_{ik} a_{ik}\, dx_i\, dx_k.$$

Consider, now, any two directions issuing from a point P of this space; these determine a plane pencil of directions or an *orientation* V_2 in V_n, and the geodesics of V_n passing through P in these directions determine a spread of two dimensions, called a *geodesic surface.* The Gaussian curvature of this geodesic surface is defined as the Riemannian curvature of the space V_n *relative to the above orientation* V_2. Thus, in general, the Riemannian curvature of a space V_n varies with the assigned orientation through a point P.

Now in defining the above concept, we have used the geodesics of the space, defined as usual as the curves which render

$$\delta \int ds = 0,$$

(where δ is the ordinary symbol of variation). We may generalize the concept of the geodesics of V_n by defining a system of curves in V_n which render

$$\delta \int F(x_1, x_2, \ldots, x_n)\, ds = \delta \int e^{\phi(x_1, x_2, \ldots, x_n)}\, ds = 0,$$

where, as indicated, F or ϕ is an arbitrary function of the coördinates only, (e being the Napierian base of logarithms). We shall call these

curves the *trajectories* of the space.[1] A point and a direction of V_n determine uniquely one *trajectory*. As above, *any two directions through a point P determine a plane pencil of directions or an orientation V_2 in V_n, and the trajectories of V_n passing through P in these directions determine a surface or spread of two dimensions, which we shall call a trajectory surface.* The geodesic surfaces are the trajectory surfaces for which F or ϕ reduces to a constant. The function ϕ determines uniquely a vector at each point of space, which we shall call the *trajectory vector*.

We shall first study the Gaussian curvatures of the trajectory surfaces through a point P, and the variation of these curvatures with the variation of the assigned orientation through P(§2). Defining as *corresponding surfaces*, the geodesic and the trajectory surfaces determined by the same orientation through a point P in V_n, we find that for two corresponding surfaces to have the same curvature, it is necessary and sufficient that the orientation contain the direction of the trajectory vector (§3). The curvature of a trajectory surface enters naturally into a study of the motion of a direction around a closed infinitesimal cycle by *conformal parallelism*, and thus leads to a geometric interpretation of certain invariants or differential parameters of an arbitrary function ϕ of the coördinates (§4).

A second part of the paper is concerned with a generalization of some of the work of Ricci on the *principal directions* of a Riemannian space. The generalization consists in replacing geodesics and geodesic surfaces by trajectories and trajectory surfaces. We begin by considering a space of *three* dimensions, and we find that through each point of such a space there are, in general, three mutual orthogonal directions in which the curvature of the trajectory surfaces normal to these directions, is a maximum or a minimum. We thus determine the *principal trajectory curvatures* of the space in terms of which the curvature of any other trajectory surface is expressed (§5).

Proceeding to a space of n dimensions, we define the *median trajectory curvature*, and we find that at each point of the space there are,

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[1] These curves are sometimes called "*a natural family of curves.*" We have chosen the brief description "*trajectories,*" since any such system of curves may be considered as the dynamical trajectories by the principle of least action in a conservative field of force and for a given constant of energy. The geodesics of V_n are the trajectories under a constant force.

in general, n mutually orthogonal directions in which the median trajectory curvature is a maximum or a minimum. These directions determine *n principal trajectory congruences and n principal median trajectory curvatures* or *invariants* in V_n (§6). These directions and invariants are then characterized in terms of the Ricci coefficients of rotation and the function ϕ (§7). Returning to the case where $n = 3$, we show that here the directions of principal trajectory curvature and the directions of principal median trajectory curvature coincide, and we find the equations which express the invariants of one type in terms of the invariants of the other type (§8). Finally our results are shown to be valid if we replace the trajectories by the more general system of curves called *velocity curves* (§9).

It is the purpose of the author to continue the developments here begun in papers which are to follow.

2. **The Curvature of a Trajectory Surface.** If the first fundamental form of V_n is given by [2]

(1) $$ds^2 = \sum_{ik} a_{ik}\, dx_i\, dx_k,$$

and the trajectories are defined by

(2) $$\delta \int e^{\phi} ds = 0,$$

where ϕ is a function of the coördinates only, the differential equations of the trajectories, found by an application of the ordinary methods of the calculus of variation, are [3]

(3) $$\frac{d\xi^{(i)}}{ds} + \sum_{\lambda\mu} \begin{Bmatrix} \lambda\mu \\ i \end{Bmatrix} \xi^{(\lambda)}\, \xi^{(\mu)} = \phi^{(i)} - \xi^{(i)} \sum_k \phi_k \xi^{(k)}, \qquad (i = 1, 2, \ldots, n);$$

where the $\xi^{(i)} = \dfrac{dx_i}{ds}$ are the *parameters* of any direction, i.e.

(4) $$\sum_{ik} a_{ik}\, \xi^{(i)}\, \xi^{(k)} = 1,$$

and where the $\phi_i = \dfrac{\partial \phi}{\partial x_i}$ form a covariant system determining the

[2] All the summations are to extend from 1 to n for the indicated subscripts unless otherwise specified.

[3] See the author's paper, *Some geometric investigations on the general problem of dynamics*, Proc. Am. Acad. of Arts and Sciences, Vol. 55 (1920), pp. 283–322.

direction of the trajectory vector ϕ, and the $\phi^{(i)}$ are the corresponding contravariant system; these systems are related by

$$(5)\qquad \phi^{(i)} = \sum_k a^{(ik)}\phi_k; \qquad \phi_i = \sum_k a_{ik}\,\phi^{(k)},$$

the $a^{(ik)}$ being the coefficients of the fundamental form reciprocal to (1). Further, the $\left\{ {\lambda\mu \atop i} \right\}$ are the Christoffel symbols of the second kind and between these and the $\left[{\lambda\mu \atop i} \right]$, the symbols of the first kind, we have the relations

$$(6)\qquad \left\{ {\lambda\mu \atop i} \right\} = \sum_k a^{(ik)} \left[{\lambda\mu \atop k} \right]; \qquad \left[{\lambda\mu \atop i} \right] = \sum_k a_{ik} \left\{ {\lambda\mu \atop k} \right\}.$$

Consider a point P_0 with coördinates x_i^0 and two directions with parameters $\xi_1^{(i)}$ and $\xi_2^{(i)}$ issuing from P_0. These determine a plane pencil of directions or orientation V_2, and the trajectories of V_n passing out from P_0 in these directions form a trajectory surface σ with pole P_0. Any direction in the pencil has parameters

$$(7)\qquad \zeta^{(i)} = \alpha\,\xi_1^{(i)} + \beta\,\xi_2^{(i)}.$$

Following the work of Bianchi for geodesics,[4] let us develop the coördinates x_i of any point of the trajectory (3) in a direction $\zeta^{(i)}$ in powers of the arc s measured from P_0, i.e.

$$x_i = x_i^0 + \zeta^{(i)}\,s + \tfrac{1}{2}\,\frac{d\zeta^{(i)}}{ds}\,s^2 + \cdots,$$

so that along a trajectory,

$$x_i = x_i^0 + \zeta^{(i)}s + \tfrac{1}{2}\left[-\sum_{\lambda\mu} \left\{ {\lambda\mu \atop i} \right\} \zeta^{(\lambda)}\zeta^{(\mu)} + \phi^{(i)} \sum_{\lambda\mu} a_{\lambda\mu}\zeta^{(\lambda)}\zeta^{(\mu)} - \zeta^{(i)} \sum_{\lambda} \phi_\lambda\,\zeta^{(\lambda)} \right] s^2 + \cdots.$$

If we substitute here the value of $\zeta^{(i)}$ from (7) and choose $u_1 = s\alpha$, $u_2 = s\beta$ as variables on the trajectory surface σ, we get

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[4] Bianchi, *Geometria differenziale*, 2d edition, vol. 1, p. 339.

$$(8)\quad x_i = x_i^0 + \xi_1^{(i)} u_1 + \xi_2^{(i)} u_2 - \tfrac{1}{2} \sum_{\lambda\mu} \begin{Bmatrix} \lambda\mu \\ i \end{Bmatrix} (\xi_1^{(\lambda)} u_1 + \xi_2^{(\lambda)} u_2)(\xi_1^{(\mu)} u_1 + \xi_2^{(\mu)} u_2) + \tfrac{1}{2} \phi^{(i)} \sum_{\lambda\mu} a_{\lambda\mu} (\xi_1^{(\lambda)} u_1 + \xi_2^{(\lambda)} u_2)(\xi_1^{(\mu)} u_1 + \xi_2^{(\mu)} u_2) - \tfrac{1}{2} \sum_{\lambda} \phi_\lambda (\xi_1^{(i)} u_1 + \xi_2^{(i)} u_2)(\xi_1^{(\lambda)} u_1 + \xi_2^{(\lambda)} u_2) + \cdots .$$

From these we immediately have the values of the derivatives at P_0,

$$(9)\quad \left(\frac{\partial x_i}{\partial u_k}\right)^0 = \xi_k^{(i)}; \left(\frac{\partial^2 x_i}{\partial u_h \partial u_k}\right)^0 = - \sum_{\lambda\mu} \begin{Bmatrix} \lambda\mu \\ i \end{Bmatrix} \xi_h^{(\lambda)} \xi_k^{(\mu)} + \psi_i(h, k),$$

where h and k have the values 1 and 2, and where we have employed the abbreviation

$$(9')\quad \psi_i(h, k) = \phi^{(i)} \sum_{\lambda\mu} a_{\lambda\mu} \xi_h^{(\lambda)} \xi_k^{(\mu)} - \tfrac{1}{2} \sum_{\lambda} \phi_\lambda (\xi_h^{(i)} \xi_k^{(\lambda)} + \xi_k^{(i)} \xi_h^{(\lambda)}).$$

Let the fundamental form for the trajectory surface σ be given by

$$(10)\quad ds^2 = b_{11}\, du_1^2 + 2\, b_{12}\, du_1\, du_2 + b_{22}\, du_2^2,$$

where

$$(10')\quad b_{ik} = \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial x_\lambda}{\partial u_i} \frac{\partial x_\mu}{\partial u_k} \qquad (i, k = 1, 2).$$

Now the Gaussian curvature of the surface σ at P_0 or of the binary form (10) is given by

$$(11)\quad \mathrm{T} = \frac{b_{12,12}^{0}}{(b_{11} b_{22} - b_{12}^2)^0},$$

where the Riemannian four-indices symbol $b_{12,12}$ is defined by [5]

$$(12)\quad b_{12,12} = \frac{\partial}{\partial u_2} \begin{bmatrix} 11 \\ 2 \end{bmatrix}_b - \frac{\partial}{\partial u_1} \begin{bmatrix} 12 \\ 2 \end{bmatrix}_b + \sum_{lm}^{1,2} b^{(lm)} \left(\begin{bmatrix} 12 \\ m \end{bmatrix}_b \begin{bmatrix} 12 \\ l \end{bmatrix}_b - \begin{bmatrix} 11 \\ m \end{bmatrix}_b \begin{bmatrix} 22 \\ l \end{bmatrix}_b \right).$$

We shall first compute the value of this symbol at P_0.

[5] Bianchi, ibid., p. 73.

We have[6]

$$(13)\qquad \begin{bmatrix} ik \\ l \end{bmatrix}_b = \sum_{\lambda\mu\nu} \begin{bmatrix} \lambda\mu \\ \nu \end{bmatrix}_a \frac{\partial x_\lambda}{\partial u_i}\frac{\partial x_\mu}{\partial u_k}\frac{\partial x_\nu}{\partial u_l} + \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial^2 x_\lambda}{\partial u_i \partial u_k}\frac{\partial x_\mu}{\partial u_l},$$

and hence,

$$(14)\qquad \left\{ \begin{aligned} \begin{bmatrix} 11 \\ 2 \end{bmatrix}_b &= \sum_{\lambda\mu\nu} \begin{bmatrix} \lambda\mu \\ \nu \end{bmatrix}_a \frac{\partial x_\lambda}{\partial u_1}\frac{\partial x_\mu}{\partial u_1}\frac{\partial x_\nu}{\partial u_2} + \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial^2 x_\lambda}{\partial u_1^2}\frac{\partial x_\mu}{\partial u_2}, \\ \begin{bmatrix} 12 \\ 2 \end{bmatrix}_b &= \sum_{\lambda\tau\nu} \begin{bmatrix} \lambda\tau \\ \nu \end{bmatrix}_a \frac{\partial x_\lambda}{\partial u_1}\frac{\partial x_\tau}{\partial u_2}\frac{\partial x_\nu}{\partial u_2} + \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial^2 x_\lambda}{\partial u_1 \partial u_2}\frac{\partial x_\mu}{\partial u_2}. \end{aligned} \right.$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial u_2}\begin{bmatrix} 11 \\ 2 \end{bmatrix}_b - \frac{\partial}{\partial u_1}\begin{bmatrix} 12 \\ 2 \end{bmatrix}_b &= \sum_{\lambda\mu\nu\tau}\left(\frac{\partial}{\partial x_\tau}\begin{bmatrix} \lambda\mu \\ \nu \end{bmatrix}_a - \frac{\partial}{\partial x_\mu}\begin{bmatrix} \lambda\tau \\ \nu \end{bmatrix}_a\right)\frac{\partial x_\lambda}{\partial u_1}\frac{\partial x_\mu}{\partial u_1}\frac{\partial x_\nu}{\partial u_2}\frac{\partial x_\tau}{\partial u_2} \\ &+ \sum_{\lambda\mu\nu}\begin{bmatrix} \lambda\mu \\ \nu \end{bmatrix}_a \frac{\partial x_\mu}{\partial u_1}\frac{\partial x_\nu}{\partial u_2}\frac{\partial^2 x_\lambda}{\partial u_1 \partial u_2} - \sum_{\lambda\tau\nu}\begin{bmatrix} \lambda\tau \\ \nu \end{bmatrix}_a \frac{\partial x_\lambda}{\partial u_1}\frac{\partial x_\tau}{\partial u_2}\frac{\partial^2 x_\nu}{\partial u_1 \partial u_2} \\ &+ \sum_{\lambda\mu} a_{\lambda\mu}\frac{\partial^2 x_\lambda}{\partial u_1^2}\frac{\partial^2 x_\mu}{\partial u_2^2} - \sum_{\lambda\mu} a_{\lambda\mu}\frac{\partial^2 x_\lambda}{\partial u_1 \partial u_2}\frac{\partial^2 x_\mu}{\partial u_1 \partial u_2} + \sum_{\lambda\mu\nu}\begin{bmatrix} \lambda\mu \\ \nu \end{bmatrix}_a \frac{\partial x_\lambda}{\partial u_1}\frac{\partial x_\mu}{\partial u_1}\frac{\partial^2 x_\nu}{\partial u_2^2} \\ &- \sum_{\lambda\tau\nu}\begin{bmatrix} \lambda\tau \\ \nu \end{bmatrix}_a \frac{\partial x_\tau}{\partial u_2}\frac{\partial x_\nu}{\partial u_2}\frac{\partial^2 x_\lambda}{\partial u_1^2} + \sum_{\lambda\mu\nu}\frac{\partial a_{\lambda\mu}}{\partial x_\nu}\frac{\partial x_\mu}{\partial u_2}\frac{\partial x_\nu}{\partial u_2}\frac{\partial^2 x_\lambda}{\partial u_1^2} - \sum_{\lambda\mu\nu}\frac{\partial a_{\lambda\mu}}{\partial x_\nu}\frac{\partial x_\mu}{\partial u_2}\frac{\partial x_\nu}{\partial u_1}\frac{\partial^2 x_\lambda}{\partial u_1 d u_2} \end{aligned}$$

Substituting here the values of the derivatives at P_0 as given by (9), we find after lengthy reductions

$$(15)\qquad \begin{aligned} \frac{\partial}{\partial u_2}\begin{bmatrix} 11 \\ 2 \end{bmatrix}_b - \frac{\partial}{\partial u_1}\begin{bmatrix} 12 \\ 2 \end{bmatrix}_b &= \sum_{l\tau} a_{l\tau}\{\psi_\tau(2,2)\psi_l(1,1) - \psi_\tau(1,2)\psi_l(1,2)\} \\ &+ \sum_{\lambda\mu\nu\tau}\left\{\frac{\partial}{\partial x_\tau}\begin{bmatrix} \lambda\mu \\ \nu \end{bmatrix}_a - \frac{\partial}{\partial x_\mu}\begin{bmatrix} \lambda\tau \\ \nu \end{bmatrix}_a\right. \\ &+ \left.\sum_l\left(\begin{bmatrix} \mu\nu \\ l \end{bmatrix}_a \begin{Bmatrix} \lambda\tau \\ l \end{Bmatrix}_a - \begin{bmatrix} \lambda\mu \\ l \end{bmatrix}_a \begin{Bmatrix} \nu\tau \\ l \end{Bmatrix}_a\right)\right\} \xi_1^{(\lambda)}\xi_1^{(\mu)}\xi_2^{(\nu)}\xi_2^{(\tau)}. \end{aligned}$$

The first term in the right member may be expanded as follows. From (9′) we have

$$\psi_\nu(i,k) = \phi^{(\nu)}\sum_{\lambda\mu} a_{\lambda\mu}\xi_i^{(\lambda)}\xi_k^{(\mu)} - \tfrac{1}{2}\sum_\lambda \phi_\lambda(\xi_i^{(\nu)}\xi_k^{(\lambda)} + \xi_k^{(\nu)}\xi_i^{(\lambda)}),$$

[6] Bianchi, ibid., p. 335.

and since

$$(16) \qquad \sum_{\lambda\mu} a_{\lambda\mu}\,\xi_1^{(\lambda)}\,\xi_1^{(\mu)} = \sum_{\lambda\mu} a_{\lambda\mu}\,\xi_2^{(\lambda)}\,\xi_2^{(\mu)} = 1; \;\; \sum_{\lambda\mu} a_{\lambda\mu}\,\xi_1^{(\lambda)}\,\xi_2^{(\mu)} = \cos\theta,$$

where θ is the angle between the directions $\xi_1^{(i)}$ and $\xi_2^{(i)}$, we may write

$$\begin{cases} \psi_l(1,1) = \phi^{(l)} - \sum\limits_{\lambda} \phi_\lambda\,\xi_1^{(l)}\,\xi_1^{(\lambda)}, \\ \psi_\tau(2,2) = \phi^{(\tau)} - \sum\limits_{\mu} \phi_\mu\,\xi_2^{(\tau)}\,\xi_2^{(\mu)}, \\ \psi_l(1,2) = \phi^{(l)}\cos\theta - \frac{1}{2}\sum\limits_{\lambda} \phi_\lambda\,(\xi_1^{(l)}\,\xi_2^{(\lambda)} + \xi_2^{(l)}\,\xi_1^{(\lambda)}) \\ \psi_\tau(1,2) = \phi^{(\tau)}\cos\theta - \frac{1}{2}\sum\limits_{\mu} \phi_\mu\,(\xi_1^{(\tau)}\,\xi_2^{(\mu)} + \xi_2^{(\tau)}\,\xi_1^{(\mu)}) \end{cases}$$

We find, after simple reductions,

$$(17) \qquad \sum_{l\tau} a_{l\tau}\,\{\psi_\tau(2,2)\,\psi_l(1,1) - \psi_\tau(1,2)\,\psi_l(1,2)\}$$

$$= \sin^2\theta \sum_{\lambda\mu} a_{\lambda\mu}\phi^{(\lambda)}\phi^{(\mu)} - \tfrac{5}{4}\sum_{\lambda\mu} \phi_\lambda\phi_\mu\,(\xi_1^{(\lambda)}\xi_1^{(\mu)} + \xi_2^{(\lambda)}\xi_2^{(\mu)} - 2\cos\theta\,\xi_1^{(\lambda)}\xi_2^{(\mu)})$$

To complete the calculation of $b^{\,0}_{12,12}$, we still have to find the value of the second term in the right member of (12). From (13) and (9), we find

$$\begin{bmatrix} i\,k \\ l \end{bmatrix}_b^0 = \sum_{\lambda\mu} a_{\lambda\mu}\,\psi_\lambda\,(i,k)\,\xi_l^{(\mu)},$$

and hence

$$\begin{bmatrix} 11 \\ 1 \end{bmatrix}_b^0 = 0; \; \begin{bmatrix} 22 \\ 2 \end{bmatrix}_b^0 = 0; \; \begin{bmatrix} 12 \\ 2 \end{bmatrix}_b^0 = -\tfrac{1}{2}\begin{bmatrix} 22 \\ 1 \end{bmatrix}_b^0; \; \begin{bmatrix} 12 \\ 1 \end{bmatrix}_b^0 = -\tfrac{1}{2}\begin{bmatrix} 11 \\ 2 \end{bmatrix}_b^0;$$

$$\begin{bmatrix} 11 \\ 2 \end{bmatrix}_b^0 = \sum_{\mu} \phi_\mu\,\xi_2^{(\mu)} - \cos\theta \sum_{\mu} \phi_\mu\,\xi_1^{(\mu)};$$

$$\begin{bmatrix} 22 \\ 1 \end{bmatrix}_b^0 = \sum_{\mu} \phi_\mu\,\xi_1^{(\mu)} - \cos\theta \sum_{\mu} \phi_\mu\,\xi_2^{(\mu)}.$$

Further, from (10′) and (9),

$$b_{ik}^0 = \sum_{\lambda\mu} a_{\lambda\mu}\,\xi_i^{(\lambda)}\,\xi_k^{(\mu)},$$

and from (16), we have

$$b_{11}^0 = b_{22}^0 = 1,\ b_{12}^0 = \cos\theta;\ (b_{11}\, b_{22} - b_{12}^2)^0 = \sin^2\theta.$$

We finally have for the second term of the right member of (12),

$$(18)\qquad \sum_{lm}^{1,2} b^{(lm)} \left(\begin{bmatrix} 12 \\ m \end{bmatrix}_b \begin{bmatrix} 12 \\ l \end{bmatrix}_b - \begin{bmatrix} 11 \\ m \end{bmatrix}_b \begin{bmatrix} 22 \\ l \end{bmatrix}_b \right)^0$$

$$= \tfrac{1}{4} \sum_{\lambda\mu} \phi_\lambda\, \phi_\mu\, (\xi_1^{(\lambda)}\, \xi_1^{(\mu)} + \xi_2^{(\lambda)}\, \xi_2^{(\mu)} - 2\cos\theta\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)})\,.$$

Hence, combining (12), (15), (17), and (18), we finally obtain

$$(19)\quad b_{12,12}^0 = \sum_{\lambda\mu\nu\tau} \left\{ \frac{\partial}{\partial x_\tau} \begin{bmatrix} \lambda\mu \\ \nu \end{bmatrix} - \frac{\partial}{\partial x_\mu} \begin{bmatrix} \lambda\tau \\ \nu \end{bmatrix} + \sum_l \left(\begin{bmatrix} \mu\nu \\ l \end{bmatrix} \begin{Bmatrix} \lambda\tau \\ l \end{Bmatrix} - \begin{bmatrix} \lambda\mu \\ l \end{bmatrix} \begin{Bmatrix} \nu\tau \\ l \end{Bmatrix} \right) \right\} \xi_1^{(\lambda)}\, \xi_2^{(\mu)}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)}$$
$$+ \sin^2\theta \sum_{\lambda\mu} a_{\lambda\mu}\, \phi^{(\lambda)}\, \phi^{(\mu)} - \sum_{\lambda\mu} \phi_\lambda\, \phi_\mu \left(\xi_1^{(\lambda)}\, \xi_1^{(\mu)} + \xi_2^{(\lambda)}\, \xi_2^{(\mu)} - 2\cos\theta\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)} \right).$$

Introducing the Riemann symbol in V_n[7]

$$a_{\lambda\mu,\nu\tau} = \frac{\partial}{\partial x_\tau} \begin{bmatrix} \lambda\mu \\ \nu \end{bmatrix} - \frac{\partial}{\partial x_\mu} \begin{bmatrix} \lambda\tau \\ \nu \end{bmatrix} + \sum_l \left(\begin{bmatrix} \mu\nu \\ l \end{bmatrix} \begin{Bmatrix} \lambda\tau \\ l \end{Bmatrix} - \begin{bmatrix} \lambda\mu \\ l \end{bmatrix} \begin{Bmatrix} \nu\tau \\ l \end{Bmatrix} \right),$$

the first differential invariant of ϕ (or the square of the magnitude of the trajectory vector)

$$(20)\qquad \Delta\phi = \sum_{\lambda\mu} a_{\lambda\mu}\, \phi^{(\lambda)}\, \phi^{(\mu)} = \sum_{\lambda} \phi^{(\lambda)}\, \phi_\lambda = R^2\,,$$

and the value

$$(b_{11}\, b_{22} - b_{12}^2)^0 = \sin^2\theta\,,$$

we obtain from (11) for *the curvature of the trajectory surface at P_0 determined by the directions $\xi_1^{(i)}$ and $\xi_2^{(i)}$ through P_0,*

$$(21)\quad T = \frac{1}{\sin^2\theta} \sum_{\lambda\mu\nu\tau} a_{\lambda\mu,\nu\tau}\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)} + \Delta\phi$$
$$- \frac{1}{\sin^2\theta} \sum_{\lambda\mu} \phi_\lambda\, \phi_\mu\, (\xi_1^{(\lambda)}\, \xi_1^{(\mu)} + \xi_2^{(\lambda)}\, \xi_2^{(\mu)} - 2\cos\theta\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)})$$

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[7] Bianchi, ibid., p. 335.

The last term may be written more symmetrically by introducing the relations (16)

$$\sum_{\nu\tau} a_{\nu\tau}\, \xi_1^{(\nu)}\, \xi_1^{(\tau)} = 1; \;\; \sum_{\nu\tau} a_{\nu\tau}\, \xi_2^{(\nu)}\, \xi_2^{(\tau)} = 1; \;\; \sum_{\nu\tau} a_{\nu\tau}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)} = \cos\theta;$$

thus we have

$$\sum_{\lambda\mu} \phi_\lambda\, \phi_\mu\, (\xi_1^{(\lambda)}\, \xi_1^{(\mu)} + \xi_2^{(\lambda)}\, \xi_2^{(\mu)} - 2\cos\theta\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)})$$

$$= \sum_{\lambda\mu\nu\tau} (a_{\mu\tau}\, \phi_\lambda\, \phi_\nu - a_{\lambda\tau}\, \phi_\mu\, \phi_\nu + a_{\lambda\nu}\, \phi_\mu\, \phi_\tau - a_{\mu\nu}\, \phi_\lambda\, \phi_\tau)\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)}\,.$$

Let us write

$$\phi_{\lambda\mu,\nu\tau} = a_{\mu\tau}\, \phi_\lambda\, \phi_\nu - a_{\lambda\tau}\, \phi_\mu\, \phi_\nu + a_{\lambda\nu}\, \phi_\mu\, \phi_\tau - a_{\mu\nu}\, \phi_\lambda\, \phi_\tau. \tag{22}$$

The quantity $\phi_{\lambda\mu,\nu\tau}$ is evidently a covariant having properties similar to those of the Riemannian covariant $a_{\lambda\mu,\nu\tau}$, viz.,

$$\begin{cases} \phi_{\lambda\mu,\nu\tau} = \phi_{\nu\tau,\lambda\mu} = -\,\phi_{\lambda\mu,\tau\nu} = -\,\phi_{\mu\lambda,\nu\tau}; \\ \phi_{\lambda\mu,\nu\nu} = \phi_{\lambda\lambda,\nu\tau} = 0; \\ \phi_{\lambda\mu,\nu\tau} + \phi_{\lambda\nu,\tau\mu} + \phi_{\lambda\tau,\mu\nu} = 0. \end{cases} \tag{22'}$$

Let us also express $\sin^2\theta$ in terms of the fundamental coefficients a_{ik} and the fundamental directions $\xi_1^{(i)}$ and $\xi_2^{(i)}$, by means of (9) and (10); we find

$$\sin^2\theta = \sum_{\lambda\mu\nu\tau} (a_{\lambda\nu}\, a_{\mu\tau} - a_{\lambda\tau}\, a_{\mu\nu})\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)}.$$

We may finally write

$$\mathrm{T} = \frac{\sum\limits_{\lambda\mu\nu\tau} (a_{\lambda\mu,\nu\tau} - \phi_{\lambda\mu,\nu\tau})\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)}}{\sum\limits_{\lambda\mu\nu\tau} (a_{\lambda\nu}\, a_{\mu\tau} - a_{\lambda\tau}\, a_{\mu\nu})\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)}} + \sum_{\lambda\mu} a_{\lambda\mu}\, \phi^{(\lambda)} \phi^{(\mu)}, \tag{21'}$$

for *the curvature of the trajectory surface at P_0 determined by two directions $\xi_1^{(i)}$ and $\xi_2^{(i)}$ through P_0.*

3. **Corresponding Geodesic and Trajectory Surfaces.** The plane pencil of directions or orientation determined by a point P and by any two directions $\xi_1^{(i)}$ and $\xi_2^{(i)}$ through P determine a geodesic surface σ_g and a trajectory surface σ_t; we shall say that σ_g and σ_t are *corresponding surfaces.*

Now, the curvature of σ_g at P is given by [8]

$$(23) \qquad \mathrm{G} = \frac{1}{\sin^2\theta} \sum_{\lambda\mu\nu\tau} a_{\lambda\mu,\nu\tau}\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)},$$

while the curvature of σ_t is given by (21). We deduce immediately

$$(24)\ \mathrm{T} = G + \Delta\phi - \frac{1}{\sin^2\theta} \sum_{\lambda\mu} \phi_\lambda\, \phi_\mu\, (\xi_1^{(\lambda)}\, \xi_1^{(\mu)} + \xi_2^{(\lambda)}\, \xi_2^{(\mu)} - 2\cos\theta\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}).$$

Equation (24) indicates the relation between the curvatures of corresponding surfaces. By introducing the angles α and β between the trajectory vector ϕ and the fundamental directions ξ_1, ξ_2,

$$\sum_\lambda \phi_\lambda\, \xi_1^{(\lambda)} = R\cos\alpha; \quad \sum_\lambda \phi_\lambda\, \xi_2^{(\lambda)} = R\cos\beta; \quad R^2 = \Delta\phi;$$

(24) may be written in the alternate form

$$(24') \qquad \mathrm{T} = G + R^2 - \frac{R^2}{\sin^2\theta}\,[\cos^2\alpha + \cos^2\beta - 2\cos\theta\cos\alpha\cos\beta].$$

We may, without loss of generality, assume the two fundamental directions ξ_1 and ξ_2 at right angles, and (24) and (24′) take the forms

$$(25) \qquad \mathrm{T} = G + \Delta\phi - \sum_{\lambda\mu} \phi_\lambda \phi_\mu\, (\xi_1^{(\lambda)}\, \xi_1^{(\mu)} + \xi_2^{(\lambda)}\, \xi_2^{(\mu)});$$

$$(25') \qquad \mathrm{T} = G + R^2 - R^2\,(\cos^2\alpha + \cos^2\beta).$$

For the special case $n = 2$, the corresponding surfaces coincide with the space; hence $T = G$. We may verify this from (25′), for since, here, ϕ, ξ_1, and ξ_2 are directions in a same plane pencil, $\cos^2\beta = \sin^2\alpha$.

In general, for any V_n, we may ask: Under what conditions will the curvatures of corresponding geodesic and trajectory surfaces be the same? From (25′), we note that if $T = G$, we must have

$$\cos^2\alpha + \cos^2\beta = 1 \quad \text{or} \quad \cos^2\beta = \sin^2\alpha, \quad \text{or} \quad \cos\beta = \pm \sin\alpha,$$

hence, ϕ, ξ_1, and ξ_2 must lie in the same plane pencil; i.e., the trajectory vector ϕ must lie in the orientation determined by the fundamental directions. We thus have the theorem:

Corresponding geodesic and trajectory surfaces at a point P which

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[8] Bianchi, ibid., p. 343.

contain the direction of the trajectory vector have the same curvatures at P; and conversely, if corresponding geodesic and trajectory surfaces are to have the same curvature at a point P, they must contain the direction of the trajectory vector at P.

Now, the geodesic surfaces which contain the direction of the trajectory vector are the osculating geodesic surfaces of the trajectories through P.[9] Hence, we may state the above theorem thus:

The osculating geodesic surface to a trajectory at a point P and its corresponding trajectory surface have the same curvature at P.

4. **Conformal Parallelism.** In a previous paper,[10] the author defined the motion of a direction by conformal parallelism as follows: Given in a space V_n two infinitely near points P and Q and a direction $\xi^{(i)}$ issuing from P; a direction $\eta^{(i)}$ in V_n through Q is said to be conformally parallel to $\xi^{(i)}$, (1) if $\eta^{(i)}$ is a direction in the trajectory surface determined by the directions $\xi^{(i)}$ and PQ, and (2) if $\eta^{(i)}$ makes at Q the same angle with the trajectory uniquely determined by P and the direction PQ as $\xi^{(i)}$ makes with it at P. If in this definition we replace trajectory and trajectory surface by geodesic and geodesic surface, we get the definition of parallelism as given by Levi-Civita and Severi.[11] The motion by conformal parallelism in V_n is evidently the conformal representation of the motion by parallelism in a space V'_n which is conformally represented on V_n, i.e. such that their first fundamental forms are related by

$$ds' = e^{\phi}\, ds,$$

the geodesics and geodesic surfaces of V'_n being represented by the corresponding trajectories and trajectory surfaces in V_n. Now, let P' be any point in V'_n and let $\eta_1^{(i)}$ and $\eta_2^{(i)}$ be any directions through P' which determine a geodesic surface containing an infinitesimal cycle of area $\delta\sigma'$ through P'. If any direction through P' and lying in the

9 See the author's paper: *Natural families of curves in a general curved space of n dimensions.* Trans. Am. Math. Soc., Vol. 13 (1912), pp. 77–95.

10 *On conformal parallelism,* Journ. of Math. and Phys., Vol. 2 (1923), No. 3.

11 T. Levi-Civita: *Nozione di parallelismo in una varietà qualunque e conseguente specificazione geometrica della curvatura Riemanniana.* Rend. del Circ. Mat. di Palermo, t. XLII (1917).

F. Severi: *Sulla curvatura delle superficie e varietà.* Rend. del Circ. Mat. di Palermo, t. XLII (1917).

geodesic surface moves by parallelism in V_n completely around the cycle, and if $\delta\epsilon'$ is the angle between its initial and final positions, then [12]

$$(26)\qquad \frac{\delta\epsilon'}{\delta\sigma'} = \frac{1}{\sin^2\theta'} \sum_{\lambda\mu\nu\tau} a'_{\lambda\mu,\nu\tau}\, \eta_1^{(\lambda)}\, \eta_2^{(\mu)}\, \eta_1^{(\nu)}\, \eta_2^{(\tau)} = G',$$

where G' is the curvature of the geodesic surface.[13] If we transform to the conformal space V_n, the corresponding conformal direction moves by conformal parallelism around an infinitesimal cycle of area $\delta\sigma$ through the corresponding point P and lying on a trajectory surface with P as pole determined by the directions $\xi_1^{(i)}$ and $\xi_2^{(i)}$, where

$$\xi_1^{(i)} = e^{\phi}\, \eta_1^{(i)}, \quad \xi_2^{(i)} = e^{\phi}\, \eta_2^{(i)}, \quad \delta\sigma = e^{-2\phi}\, \delta\sigma'.$$

If $\delta\epsilon$ is the angle between the initial and final positions of the moving direction, then $\delta\epsilon = \delta\epsilon'$, and (26) becomes

$$\frac{\delta\epsilon}{\delta\sigma} = \frac{e^{-2\phi}}{\sin^2\theta} \sum_{\lambda\mu\nu\tau} a'_{\lambda\mu,\nu\tau}\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)}.$$

We still have to express $a'_{\lambda\mu,\nu\tau}$ in terms of the elements of V_n. For this purpose we may show that[14]

$$(27)\qquad e^{-2\phi} a'_{\lambda\mu,\nu\tau} = a_{\lambda\mu,\nu\tau} + a_{\lambda\tau}(\phi_{\mu\nu} - \phi_\mu\phi_\nu) - a_{\lambda\nu}(\phi_{\mu\tau} - \phi_\mu\phi_\tau) + a_{\mu\nu}(\phi_{\lambda\tau} - \phi_\lambda\phi_\tau) - a_{\mu\tau}(\phi_{\lambda\nu} - \phi_\lambda\phi_\nu) + (a_{\lambda\tau}a_{\mu\nu} - a_{\lambda\nu}a_{\mu\tau})\Delta\phi,$$

where, as usual, we designate by ϕ_{ik} the covariant derivative of ϕ_i, i.e.,

$$\phi_{ik} = \frac{\partial\phi_i}{\partial x_k} - \sum_l \left\{ \begin{matrix} ik \\ l \end{matrix} \right\} \phi_l.$$

Introducing (27), using the relations (16), and collecting terms, we may write

$$(28)\qquad \frac{\delta\epsilon}{\delta\sigma} = \frac{1}{\sin^2\theta} \sum_{\lambda\mu\nu\tau} a_{\lambda\mu,\nu\tau}\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}\, \xi_1^{(\nu)}\, \xi_2^{(\tau)} - \Delta\phi - \frac{1}{\sin^2\theta} \sum_{\lambda\mu} (\phi_{\lambda\mu} - \phi_\lambda\, \phi_\mu)\, (\xi_1^{(\lambda)}\, \xi_1^{(\mu)} + \xi_2^{(\lambda)}\, \xi_2^{(\mu)} - 2\cos\theta\, \xi_1^{(\lambda)}\, \xi_2^{(\mu)}).$$

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[12] See T. Levi-Civita, *Qüestions de Mecànica clàssica i relativista.* Institute D'Estudis Catalans. Secció de Ciències, Barcelona, 1922.

[13] Here $\delta\epsilon'$ is considered positive or negative according as the infinitesimal rotation necessary to bring the initial direction into the final position is in the same sense as the description of the cycle or in the opposite sense.

[14] Cf. Levi-Civita, *ds^2 einsteiniani in campi newtoniani.* III, Rendic. della R. Acc. dei Lincei, t. XXVII, p. 183.

We may further introduce the curvature T and G of the corresponding trajectory and geodesic surfaces determined by ξ_1 and ξ_2, and given by (21) and (23); then (28) becomes

$$(29)\quad \frac{\delta\epsilon}{\delta\sigma} = 2G - T - \frac{1}{\sin^2\theta}\sum_{\lambda\mu}\phi_{\lambda\mu}\,(\xi_1^{(\lambda)}\,\xi_1^{(\mu)} + \xi_2^{(\lambda)}\,\xi_2^{(\mu)} - 2\cos\theta\,\xi_1^{(\lambda)}\,\xi_2^{(\mu)}).$$

Finally, let us consider the plane pencil of directions determined by two directions ξ_1 and ξ_2 through a point P. This orientation determines two corresponding surfaces, one geodesic and the other trajectory. If a direction of the pencil moves by parallelism completely around an infinitesimal cycle of area $\delta\sigma$ on the geodesic surface, it will return to a final position in the pencil and make with the initial position an angle $\delta\bar{\epsilon}$ where $\delta\bar{\epsilon}/\delta\sigma = G$. On the other hand, if this same initial direction moves by conformal parallelism completely around an infinitesimal cycle of area $\delta\sigma$ in the trajectory surface, it will return to a final position in the pencil and make with the initial position an infinitesimal angle $\delta\epsilon$, where $\delta\epsilon/\delta\sigma$ is given by (29). If we designate by $\delta\omega$ the angle between the two final positions, i.e. $\delta\omega = \delta\bar{\epsilon} - \delta\epsilon$, and by K the difference of the curvatures, i.e. $K = G - T$, we may write (29) in the form

$$(30)\quad \frac{\delta\omega}{\delta\sigma} + K = \frac{1}{\sin^2\theta}\sum_{\lambda\mu}\phi_{\lambda\mu}\,(\xi_1^{(\lambda)}\,\xi_1^{(\mu)} + \xi_2^{(\lambda)}\,\xi_2^{(\mu)} - 2\cos\theta\,\xi_1^{(\lambda)}\,\xi_2^{(\mu)})$$

Let us define the covariant system of the fourth order

$$(31)\qquad \bar{\phi}_{\lambda\mu,\nu\tau} = a_{\mu\tau}\,\phi_{\lambda\nu} - a_{\lambda\tau}\,\phi_{\mu\nu} + a_{\lambda\nu}\,\phi_{\mu\tau} - a_{\mu\nu}\,\phi_{\lambda\tau},$$

which has properties similar to the covariant $\phi_{\lambda\mu,\nu\tau}$ defined by (22) and to the Riemannian co-variant $a_{\lambda\mu,\nu\tau}$. Using the relations (16), we may write (30) in the form

$$(30')\qquad \frac{1}{\sin^2\theta}\sum_{\lambda\mu\nu\tau}\bar{\phi}_{\lambda\mu,\nu\tau}\,\xi_1^{(\lambda)}\xi_2^{(\mu)}\xi_1^{(\nu)}\xi_2^{(\tau)} = \frac{\delta\omega}{\delta\sigma} + K,$$

which gives us *a geometric interpretation of the invariant in the left member.*

When $n = 2$, the left member of (30′) reduces to

$$\Delta_2\phi = \sum_{\lambda\mu}^{1,2} a^{(\lambda\mu)}\,\phi_{\lambda\mu}\,,$$

the *second differential parameter of* ϕ in the two-spread, and since here $K = 0$, (30′) becomes

$$\Delta_2 \phi = \frac{\delta\omega}{\delta\sigma},$$

a *geometric interpretation of the second differential parameter of a function*, (as already shown in a previous paper [15]).

5. **Maximum and Minimum Trajectory Curvatures in V_3.** For the special case $n = 3$, the expression

$$\text{(21′)} \qquad T = \frac{\sum\limits_{\lambda\mu\nu\tau} (a_{\lambda\mu,\nu\tau} - \phi_{\lambda\mu,\nu\tau})\, \xi_1^{(\lambda)} \xi_2^{(\mu)} \xi_1^{(\nu)} \xi_2^{(\tau)}}{\sum\limits_{\lambda\mu\nu\tau} (a_{\lambda\nu}\, a_{\mu\tau} - a_{\lambda\tau}\, a_{\mu\nu})\, \xi_1^{(\lambda)} \xi_2^{(\mu)} \xi_1^{(\nu)} \xi_2^{(\tau)}} + \Delta\phi$$

(all summations being carried from 1 to 3 for the indicated subscripts) for the curvature of the trajectory surface determined by two directions $\xi_1^{(i)}$ and $\xi_2^{(i)}$ at a point P may be given a much simpler form by the following considerations. We shall follow the method of Ricci in his "*Lezioni sulla teoria delle superficie*," and consider as equivalent two indices whose difference is divisible by 3. Using the properties of the covariants $a_{\lambda\mu,\nu\tau}$ and $\phi_{\lambda\mu,\nu\tau}$, viz.,

$$a_{\lambda\mu,\nu\tau} = a_{\nu\tau,\lambda\mu} = -\, a_{\mu\lambda,\nu\tau} = -\, a_{\lambda\mu,\tau\nu};\ a_{\lambda\lambda,\nu\tau} = 0;$$

$$\phi_{\lambda\mu,\nu\tau} = \phi_{\nu\tau,\lambda\mu} = -\, \phi_{\mu\lambda,\nu\tau} = -\, \phi_{\lambda\mu,\tau\nu};\ \phi_{\lambda\lambda,\nu\tau} = 0;$$

and letting in turn

$$\lambda = r + 1,\ \nu = r + 2 \quad (r = 1, 2, 3, \text{ modulo } 3)$$
$$\lambda = r + 2,\ \nu = r + 1 \quad (r = 1, 2, 3, \text{ modulo } 3),$$

and then

$$\mu = s + 1,\ \tau = s + 2 \quad (s = 1, 2, 3, \text{ modulo } 3)$$
$$\mu = s + 2,\ \tau = s + 1 \quad (s = 1, 2, 3, \text{ modulo } 3),$$

we easily find

$$\text{(32)} \qquad \sum_{\lambda\mu\nu\tau} a_{\lambda\mu,\nu\tau}\, \xi_1^{(\lambda)} \xi_2^{(\mu)} \xi_1^{(\nu)} \xi_2^{(\tau)} = \sum_{rs} a_{r+1\, r+2,\, s+1\, s+2}\, (\xi_1^{(r+1)} \xi_2^{(r+2)} - \xi_1^{(r+2)} \xi_2^{(r+1)})\, (\xi_1^{(s+1)} \xi_2^{(s+2)} - \xi_1^{(s+2)} \xi_2^{(s+1)}),$$

[15] Cf. reference in footnote 10.

and

$$(33)\quad \sum_{\lambda\mu\nu\tau} \phi_{\lambda\mu,\nu\tau}\, \xi_1^{(\lambda)} \xi_2^{(\mu)} \xi_1^{(\nu)} \xi_2^{(\tau)} = \sum_{rs} (a_{r+2\,s+2}\,\phi_{r+1}\,\phi_{s+1} - a_{r+1\,s+2}\,\phi_{r+2}\,\phi_{s+1}$$
$$+ a_{r+1\,s+1}\,\phi_{r+2}\,\phi_{s+2} - a_{r+2\,s+1}\,\phi_{r+1}\,\phi_{s+2})\,(\xi_1^{(r+1)}\xi_2^{(r+2)} - \xi_1^{(r+2)}\xi_2^{(r+1)})$$
$$(\xi_1^{(s+1)}\xi_2^{(s+2)} - \xi_1^{(s+2)}\xi_2^{(s+1)}).$$

Similarly,

$$(34)\quad \sum_{\lambda\mu\nu\tau} (a_{\lambda\nu}\, a_{\mu\tau} - a_{\lambda\tau}\, a_{\mu\nu})\, \xi_1^{(\lambda)} \xi_2^{(\mu)} \xi_1^{(\nu)} \xi_2^{(\tau)}$$
$$= \sum_{rs} (a_{r+1\,s+1}\, a_{r+2\,s+2} - a_{r+1\,s+2}\, a_{r+2}\, a_{s+1})\,(\xi_1^{(r+1)}\xi_2^{(r+2)} - \xi_1^{(r+2)}\xi_2^{(r+1)})$$
$$(\xi_1^{(s+1)}\xi_2^{(s+2)}\xi_1^{(s+2)}\xi_2^{(s+1)}).$$

We now write

$$(32')\qquad \frac{a_{r+1\,r+2,\,s+1\,s+2}}{a} = \beta^{(rs)},$$

where, as Ricci has shown, $\beta^{(rs)}$ is a symmetric contravariant of the second order, a being the value of the determinant of the fundamental coefficients a_{ik}. Further, from the properties of the determinant $|\, a_{ik} \,|$ and the reciprocal determinant $|\, a^{(ik)} \,|$, we have

$$\frac{a_{r+1\,s+1}}{a} = \begin{vmatrix} a^{rs} & a^{r\,s+2} \\ a^{r+2\,s} & a^{r+2\,s+2} \end{vmatrix}, \text{ etc.},$$

so that

$$(33')\quad a_{r+2\,s+2}\,\phi_{r+1}\,\phi_{s+1} - a_{r+1\,s+2}\,\phi_{r+2}\,\phi_{s+1} + a_{r+1\,s+1}\,\phi_{r+2}\,\phi_{s+2}$$
$$- a_{r+2\,s+1}\,\phi_{r+1}\,\phi_{s+2}$$
$$= a \sum_{pq} \phi_p\,\phi_q\,(a^{(rs)}\, a^{(pq)} - a^{(ps)}\, a^{(qr)})$$
$$= a \left[a^{(rs)} \sum_{pq} a^{(pq)}\,\phi_p\,\phi_q - \sum_{pq} a^{(ps)}\, a^{(qr)}\,\phi_p\,\phi_q \right];$$
$$\text{where } \begin{Bmatrix} p = r+1,\ r+2, \text{ mod. } 3 \\ q = s+1,\ s+2, \text{ mod. } 3 \end{Bmatrix}.$$
$$= a\,[a^{(rs)}\,\Delta\phi - \phi^{(r)}\,\phi^{(s)}].$$

Also,

$$(34')\qquad \frac{a_{r+1\,s+1}\, a_{r+2\,s+2} - a_{r+1\,s+2}\, a_{r+2\,s+1}}{a} = a^{(rs)}.$$

Introducing (32′), (33′), (34′) into (32), (33), (34) and these into (21′), we obtain

$$(35)\quad T = \frac{\sum_{rs} (\beta^{(rs)} + \phi^{(r)}\phi^{(s)}) (\xi_1^{(r+1)}\xi_2^{(r+2)} - \xi_1^{(r+2)}\xi_2^{(r+1)}) (\xi_1^{(s+1)}\xi_2^{(s+2)} - \xi_1^{(s+2)}\xi_2^{(s+1)})}{\sum_{rs} a^{(rs)} (\xi_1^{(r+1)}\xi_2^{(r+2)} - \xi_1^{(r+2)}\xi_2^{(r+1)}) (\xi_1^{(s+1)}\xi_2^{(s+2)} - \xi_1^{(s+2)}\xi_2^{(s+1)})}.$$

Let us finally introduce the direction $\zeta^{(i)}$ of the normal to the surface determined by $\xi_1^{(i)}$ and $\xi_2^{(i)}$; we have

$$(36)\qquad \zeta_r = (\xi_1^{(r+1)}\xi_2^{(r+2)} - \xi_1^{(r+2)}\xi_2^{(r+1)}) \sqrt{a},$$

and we thus find

$$(37)\qquad \mathrm{T} = \frac{\sum_{rs} (\beta^{(rs)} + \phi^{(r)}\phi^{(s)}) \zeta_r \zeta_s}{\sum_{rs} a^{(rs)} \zeta_r \zeta_s} = \frac{\sum_{rs} (\beta_{rs} + \phi_r \phi_s) \zeta^{(r)}\zeta^{(s)}}{\sum_{rs} a_{rs} \zeta^{(r)} \zeta^{(s)}},$$

(where β_{rs} is the covariant reciprocal of $\beta^{(rs)}$) for *the curvature of the trajectory surface at P normal to the direction $\zeta^{(i)}$ through P.* We shall say that the trajectory surface through P is determined by the direction of its normal $\zeta^{(i)}$.

Let us now ask for the directions $\zeta^{(i)}$ for which the normal trajectory surfaces have a maximum or minimum curvature. We must evidently have

$$\frac{\partial T}{\partial \zeta^{(r)}} = 0, \quad (r = 1, 2, 3),$$

and indicating the value of T in the required direction by ω, and applying this condition to (37), we obtain the linear system

$$(38)\qquad \sum_s (\beta_{rs} + \phi_r \phi_s - \omega\, a_{rs}) \zeta^{(s)} = 0, \quad (r = 1, 2, 3).$$

Hence ω must satisfy the determinantal equation

$$(39)\qquad | \beta_{rs} + \phi_r \phi_s - \omega\, a_{rs} | = 0.$$

It is well known that because the quantities β_{rs}, $\phi_r \phi_s$, and a_{rs} are symmetric functions with respect to r and s, equation (39) has 3 real roots, which we shall designate by $\omega_1, \omega_2, \omega_3$. If these roots are dis-

tinct each will give a single system of values of the ratios of the ζ's when substituted into (38), and thus three directions are determined. If one of the ω's is a double root, there will correspond to this value a plane pencil of directions. We may then state:

There are, in general, three directions through each point in V_3 for which the corresponding trajectory surfaces have a maximum or minimum curvature.

We shall call these directions the *principal trajectory directions* and the corresponding curvatures ω_1, ω_2, ω_3, the *principal trajectory curvatures* at a point P.

If ω_1 and ω_2 are two distinct roots of (39), and $\zeta_1^{(i)}$ and $\zeta_2^{(i)}$ are the corresponding directions, then, from (38), we have

$$\sum_s (\beta_{rs} + \phi_r \phi_s) \zeta_1^{(s)} = \omega_1 \sum_s a_{rs} \zeta_1^{(s)}$$

$$\sum_r (\beta_{rs} + \phi_r \phi_s) \zeta_2^{(r)} = \omega_2 \sum_r a_{rs} \zeta_2^{(r)}$$

Multiplying the first of these by $\zeta_2^{(r)}$ and summing with respect to r, and the second by $\zeta_1^{(s)}$ and summing with respect to s, and subtracting, we find

$$(\omega_1 - \omega_2) \sum_{rs} a_{rs} \zeta_1^{(r)} \zeta_2^{(s)} = 0,$$

and hence, since $\omega_1 \neq \omega_2$,

$$\sum_{rs} a_{rs} \zeta_1^{(r)} \zeta_2^{(s)} = 0,$$

i.e. the corresponding directions are orthogonal. Hence

The 3 principal trajectory directions at a point P in V_3 are mutually orthogonal.

Let us indicate the three principal trajectory directions by the parameters $\zeta_h^{(r)}$ $(h = 1, 2, 3)$ or by the corresponding moments $\zeta_{h/r}$ $(h = 1, 2, 3)$. Equations (38) may be written

$$\sum_s (\beta_{rs} + \phi_r \phi_s) \zeta_h^{(s)} = \omega_h \sum_s a_{rs} \zeta_h^{(s)} = \omega_h \zeta_{h/r}.$$

Multiplying by $\zeta_{h/t}$ and summing on h, we have

$$\sum_{sh} (\beta_{rs} + \phi_r \phi_s) \zeta_h^{(s)} \zeta_{h/t} = \sum \omega_h \zeta_{h/r} \zeta_{h/t},$$

and since

$$\sum_h \zeta_h^{(s)} \zeta_{h/t} = \epsilon_{st} = \begin{cases} 0, \text{ if } s \neq t \\ 1, \text{ if } s = t \end{cases},$$

we obtain

$$\beta_{rs} + \phi_r \phi_s = \sum_h \omega_h \zeta_{h/r} \zeta_{h/s}, \tag{40}$$

a canonical form for the covariant $\beta_{rs} + \phi_r \phi_s$.[16]

Let $\zeta^{(i)}$ be any direction through the point P making angles $\gamma_1, \gamma_2, \gamma_3$ with the principal trajectory directions $\zeta_1^{(i)}, \zeta_2^{(i)}, \zeta_3^{(i)}$, respectively, i.e.,

$$\cos \gamma_h = \sum_r \zeta^{(r)} \zeta_{h/r}, \; (h = 1, 2, 3) .$$

The curvature of the trajectory surface normal to this arbitrary direction $\zeta^{(i)}$ is given by (37). If we introduce into this the expression (40), we have

$$T = \sum_{rsh} \omega_h \zeta^{(r)} \zeta^{(s)} \zeta_{h/r} \zeta_{h/s} = \sum_h \omega_h \cos^2 \gamma_h .$$

But ω_1, ω_2, ω_3 are the 3 principal trajectory curvatures T_1, T_2, T_3, hence

$$T = \sum_h T_h \cos^2 \gamma_h, \tag{41}$$

which expresses the curvature of any trajectory surface at a point P in terms of the 3 principal trajectory curvatures at P.

If we consider any 3 mutually perpendicular directions through P making angles $\gamma_{h/1}, \gamma_{h/2}, \gamma_{h/3}$ with the principal direction $\zeta_h^{(i)}$, we shall have for the curvatures of the corresponding trajectory surfaces

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16 Ricci and Levi-Civita have shown that a symmetric covariant α_{rs} of the second order in any number of dimensions can be expressed in one and only one way in the form $\alpha_{rs} = \sum_h \rho_h \lambda_{h/r} \lambda_{h/s}$, where the directions $\lambda_{h/r}$ ($h = 1, 2, \ldots n$) form an orthogonal ennuple at each point of V_n, and they have designated this form by the name *canonical form.* These canonical forms play a leading rôle in the geometry of the absolute calculus. Cf., Ricci, *Sulla teoria delle linee geodetiche e dei sistemi isotermi di Liouville*, §2, Atti del R. Ist. Veneto di Science, Lettere ed Arti (1894), and also, Ricci et Levi-Civita, *Méthodes de calcul différentiel absolu et leurs applications*, Math. Annalen, Bd. LIV, (1901), Chapt. II.

$$T^k = \sum_h T_h \cos^2\gamma_{h/k} \qquad (k = 1, 2, 3)\,,$$

and summing with respect to k, we get

$$\sum_k T^k = \sum_h T_h$$

Hence, *the sum of the curvatures of any three trajectory surfaces normal to three mutually perpendicular directions through a point P is constant, i. e. independent of the three directions chosen.*

All these theorems reduce to the corresponding theorems found by Ricci by a discussion of the curvatures of *geodesic surfaces* in V_3, and where he has introduced the terms *principal directions* and *principal curvatures* at a point in V_3.[17] Our expressions reduce to those found by Ricci, if we replace ϕ by a constant. We may here derive an interesting relation between the sum of the principal trajectory curvatures and the sum of the principal curvatures at a point P. From the canonical form (40), we have

$$\sum_{rs} a^{(rs)} (\beta_{rs} + \phi_r \phi_s) = \sum_{rsh} a^{(rs)} \omega_h \zeta_{h/r} \zeta_{h/s}\,,$$

or

$$\sum_{rs} a^{(rs)} \beta_{rs} + \Delta\phi = \sum_h \omega_h\,. \tag{42}$$

On the other hand, from the corresponding canonical form for β_{rs} in terms of the principal curvatures and the principal directions, viz.,

$$\beta_{rs} = \sum_h \omega_h' \zeta_{h/r}' \zeta_{h/s}'\,,$$

we get

$$\sum_{rs} a^{(rs)} \beta_{rs} = \sum_h \omega_h'\,, \tag{43}$$

By comparison of (42) and (43) we deduce

$$\sum_h \omega_h = \sum_h \omega_h' + \Delta\phi\,. \tag{44}$$

Hence, *the sum of the principal trajectory curvatures minus the sum of the principal curvatures at a point P is equal to the square of the magnitude of the trajectory vector at P.*

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[17] Ricci, *Sui gruppi continui di movimenti in una varietà qualunque a tre dimensioni*. Memorie della Società Italiana delle Science, detta dei XL, Serie III^a, t. XII.

6. **Median Trajectory Curvature and Principal Trajectory Directions in V_n.** As usual, let us define a congruence of lines in V_n by the equations

$$\frac{dx_1}{\lambda^{(1)}} = \frac{dx_2}{\lambda^{(2)}} = \ldots\ldots = \frac{dx_n}{\lambda^{(n)}},$$

the λ's being functions of the coördinates $x_1, x_2, \ldots, x_n$. We say that the system $\lambda^{(r)}$ $(r = 1, 2, \ldots, n)$ is the *contravariant coördinate system of the congruence* and that the reciprocal λ_r, $(r = 1, 2, \ldots, n)$ is the *covariant coördinate system.* We represent by [1], [2], . . ., [n] n such congruences of lines, with $\lambda_1^{(r)}, \lambda_2^{(r)}, \ldots, \lambda_n^{(r)}$ $(r = 1, 2, \ldots, n)$ or with $\lambda_{1/r}, \lambda_{2/r}, \ldots, \lambda_{n/r}$ $(r = 1, 2, \ldots, n)$ their contravariant or covariant coördinate systems. If these congruences determine n mutual orthogonal directions at each point of V_n, they are said to form an *orthogonal ennuple.*[18] For such an ennuple, we have the identities

$$(45) \quad \left\{ \begin{array}{l} \sum\limits_r \lambda_h^{(r)} \lambda_{k/r} = \epsilon_{hk} \; ; \; \sum\limits_r \lambda_r^{(h)} \lambda_{r/k} = \epsilon_{hk} \; ; \; \epsilon_{hk} = \begin{cases} 0, \text{ if } h \neq k \\ 1, \text{ if } h = k \end{cases} \\ a_{rs} = \sum\limits_h \lambda_{h/r} \lambda_{h/s} \; ; \; a^{(rs)} = \sum\limits_h \lambda_h^{(r)} \lambda_h^{(s)} . \end{array} \right.$$

Let us now return to the expression (21) for the curvature of a trajectory surface determined by two directions $\xi_1^{(i)}$ and $\xi_2^{(i)}$. The curvature of the trajectory surface determined at the point P by the directions of the lines of two congruences [h] and [k] passing through P is evidently given by

$$(46) \quad T_{hk} = \sum_{rstu} a_{rs,tu} \lambda_h^{(r)} \lambda_k^{(s)} \lambda_h^{(t)} \lambda_k^{(u)} + \Delta\phi - \sum_{rs} \phi_r \phi_s (\lambda_h^{(r)} \lambda_h^{(s)} + \lambda_k^{(r)} \lambda_k^{(s)}),$$

since $\cos\theta = 0$, $\sin\theta = 1$.

If we allow k to take all values from 1 to n, except $k = h$, i. e. if we associate with the direction of the congruence [h] the directions of $n-1$ orthogonal congruences at P, and sum with respect to k the corresponding expressions (46), we obtain

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18 Cf. Ricci et Levi-Civita, reference in footnote 16. Also, Ricci, *Dei sistemi di congruenze ortogonali in una varietà qualunque.* Memorie della R. Acc. dei Lincei, Classe dei Scienze, t. II (1896).

$$\sum_k{}' T_{hk} = \sum_{rstu} a_{rs,tu} \lambda_h^{(r)} \lambda_h^{(t)} \left(\sum_k{}' \lambda_k^{(s)} \lambda_k^{(u)} \right) + (n-1) \Delta\phi$$
$$- (n-1) \sum_{rs} \phi_r \phi_s \lambda_h^{(r)} \lambda_h^{(s)} - \sum_{rs} \phi_r \phi_s \left(\sum_k{}' \lambda_k^{(r)} \lambda_k^{(s)} \right),$$

where $\sum'$ indicates that in the summation k takes all values except the value h. If to the right member we add

$$\sum_{rstu} a_{rs,tu} \lambda_h^{(r)} \lambda_h^{(t)} \lambda_h^{(s)} \lambda_h^{(u)} + \sum_{rs} \phi_r \phi_s \lambda_h^{(r)} \lambda_h^{(s)} - \sum_{rs} \phi_r \phi_s \lambda_h^{(r)} \lambda_h^{(s)},$$

which vanishes identically, we may write

$$\sum_k{}' T_{hk} = \sum_{rstu} a_{rs,tu} \lambda_h^{(r)} \lambda_h^{(t)} \left(\sum_k \lambda_k^{(s)} \lambda_k^{(u)} \right) + (n-1) \Delta\phi$$
$$- (n-2) \sum_{rs} \phi_r \phi_s \lambda_h^{(r)} \lambda_h^{(s)} - \sum_{rs} \phi_r \phi_s \left(\sum_k \lambda_k^{(r)} \lambda_k^{(s)} \right).$$

Using the last identity (45) and introducing the covariant system

$$\alpha_{rs} = \sum_{pq} a^{(pq)} a_{pr,qs}, \tag{47}$$

we finally get, after a change of indices,

$$\sum_k{}' T_{hk} = \sum_{rs} [\alpha_{rs} - (n-2) \phi_r \phi_s] \lambda_h^{(r)} \lambda_h^{(s)} + (n-2) \Delta\phi, \tag{48}$$

which is independent of k. We thus have the result:

If at a point P and to a direction $[h]$ in V_n we associate any $(n-1)$ other directions which form an orthogonal ennuple with it, the sum of the curvatures of the trajectory surfaces determined by the direction $[h]$ and each of the $(n-1)$ associated directions, is constant, i.e., *independent of the particular $(n-1)$ associated directions considered. This sum we shall designate by the term median trajectory curvature of V_n at the point P and in the direction $[h]$.*

If we sum (48) on h, we get

$$\sum_{hk} T_{hk} = \sum_{rs} [\alpha_{rs} - (n-2) \phi_r \phi_s] \cdot \sum_h \lambda_h^{(r)} \lambda_h^{(s)} + n(n-2) \Delta\phi,$$

or

$$\sum_{hk} T_{hk} = \sum_{rs} a^{(rs)} \alpha_{rs} + (n-1)(n-2) \Delta\phi. \tag{49}$$

The right member is a function of the coördinates of P only, and we may state the result:

The sum of the median trajectory curvatures associated with an arbitrary orthogonal ennuple at a point P is constant, i.e. *independent of the particular ennuple chosen.*

It is interesting to note that since the sum of the median curvatures of the corresponding geodesic surfaces at a point P is given by

$$\sum_{hk} G_{hk} = \sum_{rs} a^{(rs)} \alpha_{rs},$$

we have the relation

$$\sum_{hk} T_{hk} - \sum_{hk} G_{hk} = (n-1)(n-2)\,\Delta\phi. \tag{50}$$

Let us now ask for the directions $[h]$ through a point P for which the corresponding trajectory median curvature is a maximum or a minimum. Writing (48) in the form

$$\sum_{k} T_{hk} = \frac{\sum_{rs} [\alpha_{rs} - (n-2)\phi_r\phi_s]\,\lambda_h^{(r)}\lambda_h^{s}}{\sum_{rs} a_{rs}\lambda_h^{(r)}\lambda_h^{(s)}} + (n-2)\,\Delta\phi,$$

we note that the last term $(n-2)\,\Delta\phi$ is independent of the direction. Indicating by ρ the value of $\sum_k T_{hk}$ in the required direction, we have for the maximum or minimum condition, (as in §5),

$$\sum_{s} [\alpha_{rs} + (n-2)(a_{rs}\Delta\phi - \phi_r\phi_s) - \rho a_{rs}]\,\lambda_h^{(s)} = 0, \quad (r = 1, 2, \ldots, n); \tag{51}$$

hence, the ρ's must satisfy the determinantal equation

$$|\alpha_{rs} + (n-2)(a_{rs}\Delta\phi - \phi_r\phi_s) - \rho a_{rs}| = 0. \tag{52}$$

An equation of this form has, as is well known, n real roots, in general distinct, which when substituted into (51) determine n directions through the point P and, as in §5, it is a simple matter to show that these n directions are mutually orthogonal. We thus have the theorem:

At each point of V_n, there exist, in general, n mutually orthogonal directions corresponding to which the median trajectory curvature is a maximum or a minimum.

Analogous to the work of Ricci,[19] we shall name the directions thus determined at each point P, the *principal trajectory directions,* corresponding ennuples the *principal trajectory ennuples* or *principal trajectory congruences,* and the invariants ρ_h $(h=1, 2, \ldots, n)$, the *principal trajectory invariants* or *principal trajectory median curvatures* of the space V_n.

From (51), we immediately deduce

$$\alpha_{rs} + (n-2)\,(a_{rs}\Delta\phi - \phi_r\,\phi_s) = \sum_h \rho_h\,\lambda_{h/r}\,\lambda_{h/s}, \tag{52}$$

the *canonical form for the symmetric covariant* $\alpha_{rs}+(n-2)\,(a_{rs}\Delta\phi-\phi_r\phi_s)$.

The corresponding canonical form which determines the principal invariants of Ricci, ρ_h' $(h = 1, 2, \ldots, n)$ is

$$\alpha_{rs} = \sum_h \rho_h'\,\lambda_{h/r}'\,\lambda_{h/s}'. \tag{53}$$

Multiplying (52) by $a^{(rs)}$ and summing with respect to r and s, we obtain

$$\sum_{rs} a^{(rs)}\,\alpha_{rs} + (n-1)\,(n-2)\,\Delta\phi = \sum_h \rho_h.$$

Similarly, from (53) we obtain

$$\sum_{rs} a^{(rs)}\,\alpha_{rs} = \sum_h \rho_h'.$$

By comparison,

$$\sum_h \rho_h = \sum_h \rho_h' + (n-1)\,(n-2)\,\Delta\phi. \tag{54}$$

This gives *the relation between the sums of the principal invariants and the principal trajectory invariants at a point* P *in* V_n.

7. **Characterization of the Principal Trajectory Invariants and Directions.** We may characterize the principal trajectory invariants and directions as follows. Ricci has defined the *coefficients* of *rotation*[20] in terms of the directions of an orthogonal ennuple [1], [2], ..., [n] as

$$\gamma_{hij} = \sum_{rs} \lambda_{h/rs}\,\lambda_i^{(r)}\lambda_j^{(s)},$$

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[19] *Direzione e invarianti principali in una varietà qualunque.* Atti del R. Ist. Veneto di Scienze, Lettere, ed Arti, t. LXIII, (1903–4).
[20] Cf. reference in footnote 19.

where $\lambda_{h/rs}$ is the covariant derivative of $\lambda_{h/r}$ with respect to s. From these we form the derived quantities

$$\gamma_{hi,kl} = \frac{\partial \gamma_{hik}}{\partial s_l} - \frac{\partial \gamma_{hil}}{\partial s_k} + \sum_j \{\gamma_{hij} (\gamma_{jkl} - \gamma_{jlk}) + \gamma_{jhl}\gamma_{jik} - \gamma_{jhk}\gamma_{jil}\}.$$

From these it may be shown that

$$\gamma_{hi,kl} = \sum_{rstu} a_{rs,tu} \lambda_h^{(r)} \lambda_i^{(s)} \lambda_k^{(t)} \lambda_l^{(u)},$$

or, by solving for the Riemann symbols,

$$a_{pr,qs} = \sum_{hikl} \gamma_{hi,kl} \lambda_{h/p} \lambda_{i/r} \lambda_{k/q} \lambda_{l/s}.$$

Applying this to (47) we may write

$$\alpha_{rs} = \sum_{pq} a^{(pq)} a_{pr,qs} = \sum_{hil} \gamma_{hi,hl} \lambda_{i/r} \lambda_{l/s}. \tag{55}$$

Now if β_i is the angle between the trajectory vector $\phi^{(r)}$ and the congruence $[i]$, then

$$\cos\beta_i = \frac{1}{\sqrt{\Delta\phi}} \sum_r \phi_r \lambda_i^{(r)},$$

or

$$\phi_r = \sqrt{\Delta\phi} \sum_i \cos\beta_i \lambda_{i/r},$$

and hence,

$$\phi_r \phi_s = \Delta\phi \sum_{il} \cos\beta_i \cos\beta_l \lambda_{i/r} \lambda_{l/s}. \tag{56}$$

Furthermore,

$$a_{rs} = \sum_i \lambda_{i/r} \lambda_{i/s}.$$

Hence,

$$\begin{aligned} (57)\quad & \alpha_{rs} + (n-2)(a_{rs}\Delta\phi - \phi_r \phi_s) = \\ & \sum_i \left[\sum_h \gamma_{hi,hi} + (n-2)\Delta\phi \sin^2\beta_i \right] \lambda_{i/r} \lambda_{i/s} \\ & \quad + \sum_{il}^{i \neq l} \left[\sum_h \gamma_{hi,hl} - (n-2)\Delta\phi \cos\beta_i \cos\beta \right] \lambda_{i/r} \lambda_{l/s}. \end{aligned}$$

Comparing this with the canonical form for the left member as given by (52), we find

$$(58)\qquad \sum_h \gamma_{hi,hi} + (n-2)\,\Delta\,\phi\,\sin^2\beta_i = \rho_i\,,$$

which define the principal trajectory invariants ρ_i, and

$$(59)\qquad \sum_h \gamma_{hi,hl} - (n-2)\,\Delta\phi\,\cos\beta_i\,\cos\beta_l = 0, \quad (i \neq l),$$

which serve to characterize the principal trajectory ennuples of any V_n.

8. **Principal Trajectory Directions when n = 3.** In § 5, we derived, for the case $n = 3$, certain relations by means of a direct discussion of the curvatures of trajectory surfaces at a point P, whereas in § 6, we derived similar relations by means of a discussion of the median trajectory curvature at a point P. We may ask how the directions and invariants found in these two discussions are related.

For this purpose, let us study the canonical forms

$$(40)\qquad \beta_{rs} + \phi_r\,\phi_s = \sum_h \omega_h\,\zeta_{h/r}\,\zeta_{h/s}\,,$$

and

$$(52)\qquad \alpha_{rs} + (a_{rs}\,\Delta\,\phi - \phi_r\,\phi_s) = \sum_h \rho_h\,\lambda_{h/r}\,\lambda_{h/s}\,,$$

which serve to characterize the two sets of directions. Here the $\beta^{(rs)}$ are defined by

$$(32')\qquad \beta^{(rs)} = \frac{a_{r+1\,r+2,\,s+1\,s+2}}{a},$$

and the α_{rs} by

$$(47)\qquad \alpha_{rs} = \sum_{pq} a^{(pq)}\,a_{pr,qs}\,,$$

(all summations being carried from 1 to 3 for the indicated subscripts). Using the properties of the Riemann symbol, we have

$$\begin{aligned}\alpha_{rs} = {} & a^{(r+1\,s+1)}\,a_{r+1\,r,\,s+1\,s} + a^{(r+2\,s+1)}\,a_{r+2\,r,\,s+1\,s} + \\ & \qquad a^{(r+1\,s+2)}\,a_{r+1\,r,\,s+2\,s} + a^{(r+2\,s+2)}\,a_{r+2\,r,\,s+2\,s} \\ = {} & a[a^{(r+1\,s+1)}\,\beta^{(r+2\,s+1)} - a^{(r+2\,s+1)}\,\beta^{(r+1\,s+2)} - a^{(r+1\,s+2)}\,\beta^{(r+2\,s+1)} + \\ & \qquad a^{(r+2\,s+2)}\,\beta^{(r+1\,s+1)}].\end{aligned}$$

But

$$a\,.\,a^{(r+1\,s+1)} = \begin{vmatrix} a_{rs} & a_{r\,s+2} \\ a_{r+2\,s} & a_{r+2\,s+2} \end{vmatrix}, \text{ etc.}$$

Hence

$$\alpha_{rs} = \sum_{pq} \beta^{(pq)} (a_{rs}\, a_{pq} - a_{ps}\, a_{qr}), \text{ where} \begin{cases} p = r+1,\ r+2, \text{ mod. } 3 \\ q = s+1,\ s+2, \text{ mod. } 3 \end{cases}$$

$$= a_{rs} \sum_{pq} a_{pq}\, \beta^{pq} - \sum_{pq} a_{ps}\, a_{qr}\, \beta^{(pq)}$$

$$= a_{rs} \sum_{pq} a^{(pq)}\, \beta_{pq} - \beta_{rs}.$$

Now, we have shown, by (42), that

$$\sum_{pq} a^{(pq)}\, \beta_{pq} + \Delta\, \phi = \sum_{h} \omega_h = L,$$

which is independent of the directions through P; therefore

$$\alpha_{rs} = a_{rs}\, (L - \Delta\phi) - \beta_{rs}.$$

But

$$a_{rs} = \sum_{h} \zeta_{h/r}\, \zeta_{h/s}\,,$$

hence,

$$\beta_{rs} = \sum_{h} (L - \Delta\phi)\, \zeta_{h/r}\, \zeta_{h/s} - \alpha_{rs};$$

substituting this into (40), we have the equivalent canonical form for determining the invariants ω,

$$(40') \qquad \alpha_{rs} + (a_{rs}\, \Delta\, \phi - \phi_r\, \phi_s) = \sum_{h} (L - \omega_h)\, \zeta_{h/r}\, \zeta_{h/s}.$$

Finally, comparing the two canonical forms (40′) and (52) for the same expression $\alpha_{rs} + (a_{rs}\, \Delta\phi - \phi_r\, \phi_s)$, we deduce that the directions $\zeta^{(i)}_{h}$ coincide with the directions $\lambda^{(i)}_{h}$ and that the invariants ω_h and ρ_h are related by

$$(60) \qquad \rho_h = \omega_1 + \omega_2 + \omega_3 - \omega_h, \qquad (h = 1, 2, 3).$$

This may be written in the equivalent forms

$$(61) \qquad \begin{cases} \rho_h = \omega_{h+1} + \omega_{h+2} \\ 2\omega_h = \rho_{h+1} + \rho_{h+2} - \rho_h \end{cases}, \ (h = 1, 2, 3)$$

We may then state the result:

In any space of 3 dimensions, the principal trajectory directions defined by means of the principal trajectory curvatures and by means of the principal trajectory median curvatures coincide for every point of the space; in other words, the principal trajectory congruences thus deter-

mined coincide. The relations between the two sets of invariants ω_h *(the principal trajectory curvatures) and* ρ_h *(the principal trajectory median curvatures) are given by* (61).

It is interesting to note that due to the absence of the function ϕ from equations (61), the relations between the principal trajectory invariants ρ and ω have exactly the same form as the relations between the principal invariants ρ' and ω'.

9. **A Generalization. Velocity Systems.** We note that if in equations (3), § 2,

$$(3) \qquad \frac{d\xi^{(i)}}{ds} + \sum_{\lambda\mu} \left\{ {\lambda\mu \atop i} \right\} \xi^{(\lambda)} \xi^{(\mu)} = \phi^{(i)} - \xi^{(i)} \sum_k \phi_k \xi^{(k)}, \quad (i = 1, 2, \ldots, n),$$

the quantities ϕ_i are not taken as the partial derivatives of a function ϕ with respect to the coördinates, but as components of an arbitrary point vector, these equations no longer represent a system of curves defined by a variation problem of the form

$$(2) \qquad \delta \int e^{\phi}\, ds = 0.$$

In this case the curves are said to form a *velocity system*, and a dynamical interpretation of these is found in the following statement:— A curve is a velocity curve corresponding to a given constant speed c, if a particle starting from any point of such a curve in the direction of the curve and with that speed, describes a trajectory, in a field under any positional forces, osculating the curve; in the above equations, these forces have as components the quantities $c^2\phi_i$.[21]

Now, all our results of §§ 2, 3, 5–8 were derived without the assumption that $\phi_i = \dfrac{\partial \phi}{\partial x_i}$ $(i = 1, 2, \ldots, n)$, so that all these results are valid if trajectories and trajectory surfaces are replaced by velocity curves and their corresponding surfaces.

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[21] Cf. the author's *Note on velocity systems in a curved space of N dimensions*, Bull. Am. Math. Soc., Vol. 27, 1920, pp. 71–77.
For a geometric characterization of velocity systems see the author's paper referred to in footnote 9.

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